Singularities in turbulent flows: how to observe them ?

Martine Le Berre[1], Thierry Lehner[2] and Yves Pomeau[1]
[1] *Laboratoire d'Hydrodynamique, Ladhyx, (CNRS UMR 7646), Ecole Polytechnique, 91128 Palaiseau, France*
[2]*LUTH, UMR 8102 CNRS, Observatoire de Paris- Meudon, 5 Place J. Janssen, 92195 Meudon, France.*
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The existence and relevance of finite time singularities of the fluid equations are outstanding problems in turbulent flows at high Reynolds number. Their existence is based on recent analytical studies of the time dependent equations for inviscid flow and it explains well the long known observation of intermittency there. The analysis of hot wire records of a turbulent flow in the wind tunnel of Modane is in fair agreement with this existence of finite time singularities passing by the point of recording. This analysis relies on the behavior of the structure functions of the Eulerian acceleration at increasing exponents. It is completed by the analysis of Lagrangian data coming from a von Karman experiment. A particular choice of multi-correlation functions, called test functions, makes them sensitive to the irreversible character of the dynamics, which is shown to be effective at short time only, that we interpret as a signature of the presence of singular events developing a very large acceleration on a very short time scale, of the order of the Kolmogorov dissipation time.

I. INTRODUCTION

The idea of existence of singularities in incompressible 3D flows at high Reynolds number and finite energy has a fairly long history, going back at least to the publication by Jean Leray in 1934 [1] where he wrote the equations for self-similar singular solutions of the Navier-Stokes equations without deciding if they have or not non trivial solutions, a question that remains unanswered after almost ninety years. However it is fair to say that mathematical results point [2] to the non-existence of self-similar solutions of the equations as written by Leray for Navier-Stokes equations. But recent results point to the existence of self similar solutions of the inviscid Euler equations showing a finite time blow-up for axisymmetric geometries[3]-[4] . Another early mention of finite time singularities in fluids is in a 1939 paper by G.I. Taylor [5] where the author states "The dissipation of energy is due chiefly to the formation of small regions where the vorticity is very high". From Taylor's paper it is hard to see wherefrom he got this intuition and what he meant exactly, but it is not a huge extrapolation to say that he had in mind the occurence of singularities and that these events explain the large dissipation as well as the intermittency observed in turbulent flows. It has been argued too in a recent paper[6] that singularities are necessary to explain Newton's quadratic friction law on blunt bodies moving fastly in a fluid. This law tells that the drag on such a body is proportional to $|U|U$ where U is the speed of the relative motion. The absolute value and the lack of any parameter besides the geometrical parameters of the body shows that this *non analytic* Newton's law of drag cannot be explained as the result of smooth solutions of the time dependent Euler equations. In summary the solution of the Euler equations relevant for computing Newton's drag must be non analytic and/or singular in some sense to yield the absolute value of the velocity in the friction law.

Therefore it is of outstanding interest to look at ways of putting in evidence finite time singularities in real turbulent flows. This meets obvious challenges: suppose, as implied both by Leray and by Taylor, that singular solutions of the Euler equations are formed at some points in space and time. In real conditions, they don't reach the blooming stage, but likely become smoothed by viscous friction. Their locality together with their final smooth stage makes it very hard to record them with standard hot wire probes for instance. Such a probe records the velocity field at a given point as a function of time where it has virtually no chance to be hit by such a punctual event in a 3D space. Secondly, even if such a record was possible, it would remain to put in evidence that dissipation of energy takes place there, as implied by Taylor. Therefore putting in evidence singularities is somehow not sufficient, one should also put in evidence that dissipation occurs there. Note that it is also well possible that such a singularity is not dissipative in itself and is just like the bounce of an elastic ball, where the initial kinetic energy is restored after the velocity changes sign. All this leads to two questions: first is it possible to show the existence of singularities in flows at high Reynolds number on hot wire records at a single point, or by other techniques as acoustic methods tracking the motion of tracer particles? Then is it possible to show in the same conditions that dissipation takes place there?

In a previous work[7] we have investigated whether Leray-type singular solutions which blow up at time t^* on a point, of the form

$$\mathbf{u}(\mathbf{r},t) = \frac{1}{(t^*-t)^\alpha}\mathbf{U}(\frac{\mathbf{r}}{(t^*-t)^{1-\alpha}}), \tag{1}$$

are compatible with the hot wire records taken in the wind tunnel of Modane at high Reynolds number. Studying the scaling laws between the large accelerations and velocities, we have found that the data are compatible with the value $\alpha = 1/2$ which ensures the conservation of the circulation around the the singularity, however up to the uncertainty of the measures, the experimental results are also compatible with the Sedov-Taylor value $\alpha = 3/5$ which ensures the conservation of energy in the singular domain. Nevertheless it likely happens that many other finite time singular solutions co-exist in the turbulent flow as the ones

predicted in Refs.[3]-[4]. Such axi-symmetric singularities blow-up on a line (not on a point) with other values of the exponents, that lead us to investigate their existence rather than fitting the data with some analytical expression.

Recently two of us found a way to observe indirectly finite time singularities in those records, as detailed in [8]. This was done by investigating the structure functions of the acceleration, which were rarely investigated although they are much more sensitive to large fluctuations than the usual structure functions of the velocity. In sect. II we present the enlightening results of this study which is definitely in favor of the existence of singular events in the Modane's flow. In this case, the acceleration is the time derivative of the velocity recorded at a given point (the hot wire place), sometimes called Eulerian acceleration. In addition we show that similar results are obtained by investigating the Lagrangian data in a von Karman set-up[9].

Finally we introduce in Sec.III a set of multi-time correlation functions, called test functions, which proves the irreversible character of this flow and depicts dissipation on a short time scale only, of the order of the Kolmogorov dissipation time.

II. STRUCTURE FUNCTIONS OF THE ACCELERATION

Our starting point is to explain the transition of shape of the structure functions, see Eq.(3), which occurs at short distance (or time) when the order n of the exponent varies.

A. Eulerian acceleration

As said before we shall consider structure functions computed with what we call the Eulerian acceleration measured from hot-wire records[10] of turbulence taken in the wind tunnel of Modane. We define this acceleration as the difference between two successive values of the measured velocity,

$$a(t) = \frac{v(t_{i+1}) - v(t_i)}{t_u} \tag{2}$$

where t_i are the recording times separated by the sampling time $t_u = t_{i+1} - t_i$. This simple time difference is a priori the closest one to the time derivative although we must notice that this finite difference includes also second derivatives and higher derivatives when expanded in Taylor series of the time difference. We neglect the higher order contributions to keep the mathematical handling as simple as possible.

From the data we compute "time" "structure functions

$$S_n(\tau) = < \left(a(t+\tau) - a(t)\right)^n > , \tag{3}$$

by gliding average as explained in Ref.[11], and look at the behavior of this function (versus τ) as the exponent n increases. As shown in figure 1-a, the set of structure functions $S_{2n}(\tau)$ display a dramatic change of behavior at τ small as n increases beyond a critical value of order 4. For $n < 4$ the structure function tends smoothly to zero as τ decreases to zero, but as n becomes bigger than 4 the structure function displays a sort of "numerical" divergence as τ decreases, with a peak becoming higher and higher (and steeper and steeper) as the order n (of the S_{2n} functions) increases. The peak is located at $\tau = 2t_u$ namely for a time interval τ close to the Kolmogorov time t_K (see data in the cations). This is, we believe, a convincing way to show that finite time singularities of the Leray type do occur in turbulent flows at very large Reynolds number. One notice that similar behavior is observed in Fig.b for $S_{2n+1}(\tau)$, although the curves are more chaotic (this is probably related to the small values of $S_{2n+1}(\infty)$ as compared to $(S_{2n}(\infty) - S_{2n+2}(\infty))/2$).

We explain that the short time behavior of the structure functions of Eulerian acceleration results from singularities passing nearby the point of measurement. To be more specific, the probability that a single point fluctuation occurs at distance less than r from the point of measurement, is proportional to $\frac{4\pi}{3}\rho r^3$ where ρ is the number density of the singular points. Therefore by recording the fluctuations, the influence of finite amplitude fluctuations will be vanishing at small r. This effect should decrease like r^{-3}.

Let us assume that instead of a point there is a local field (like the Eulerian acceleration) depending on the distance to the location of the singularity with a negative power law, say $r^{-\beta}$ with β positive. Such a power law could be the one of self similar singular solutions of the Leray -type (1) where $\beta = 1 - \alpha$. The small probability of being close to the location of the singularity can be overbalanced in the structure function by the amplitude of the fluctuation raised at a big enough power, a power depending on n that can be chosen at will. If the power of the fluctuation is p, and if $p\beta > 3$ then instead of observing contribution to the local amplitude decaying like r^3 at small r, one will observe a structure function growing to infinity at short distances like $r^{3-p\beta}$. Such a sharp transition in short distance behavior of structure function has been observed both in hot wire records in Modane wind tunnel and also in numerical studies of the focusing non linear Schrödinger equation [12].

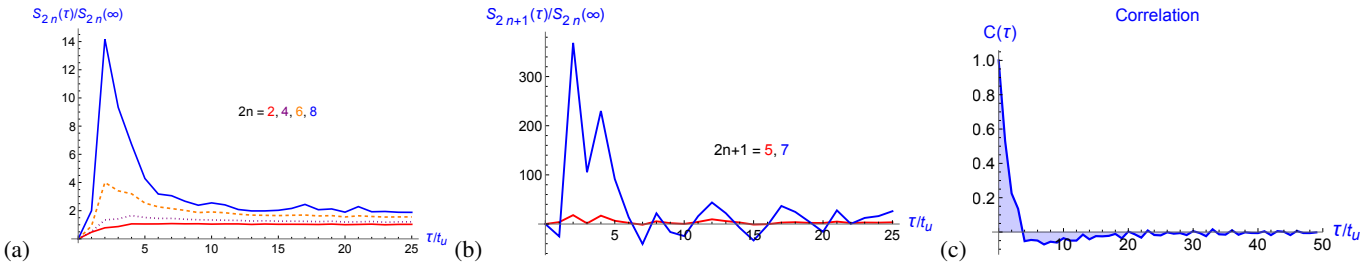


FIG. 1: (a) Structure functions $S_{2n}(\tau)/S_{2n}(\infty)$ of the Eulerian acceleration scaled to the asymptotic value at large τ, versus τ/t_u from Modane's experiment. (b) Idem for even order structures functions. For $\tau = 0$ the structure functions vanish by definition, see Eqs. (2)-(3). (c) Correlation function of the acceleration $S_1(\tau)/ < a^2 >$, defined by Eq.(3) versus τ/t_u. The Reynolds number of the turbulent flow is $Re_\lambda = 2500$, $t_u = 0.04ms$, smaller than the Kolmogorov time $t_K = 0.1ms$. The velocity is $20m/s$ in average, with r.m.s $\sigma_v = 1.67m/s$. With a sampling frequency of $25Khz$, and about 10 minutes of record, the number of sampling points is about $14 \cdot 10^6$

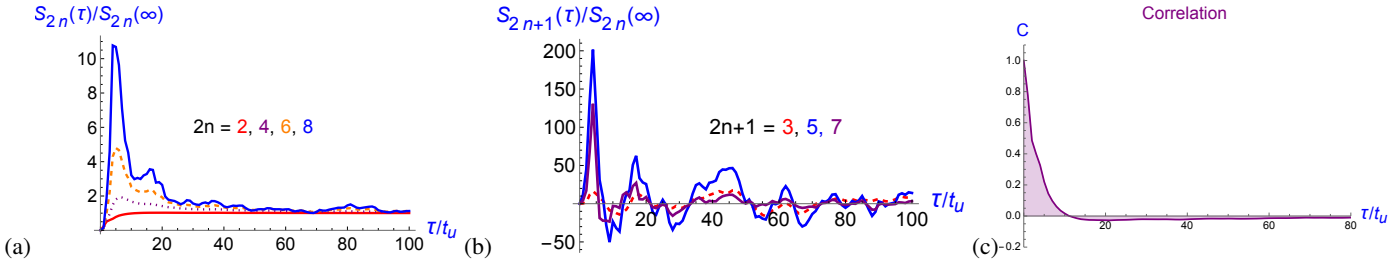


FIG. 2: von Karman experiment. (a) Structure functions $S_{2n}(\tau)/S_{2n}(\infty)$ of the Lagrangian acceleration scaled to the asymptotic value at large τ, versus τ/t_u. (b) Idem for even order structures functions. For $\tau = 0$ the structure functions vanish by definition, see Eqs. (2)-(3). (c) Correlation function of the Lagrangian acceleration $S_1(\tau)/ < a^2 >$, defined by Eq.(3) versus τ/t_u, with acceleration given by Eq.(4). $Re_\lambda = 740$, sampling time $t_u = 0.15ms$, dissipative time scale $t_k = 0.2ms$, diameter of the ball $0.25mm$, cut-off time scale due to the ball diameter $0.4ms$, correlation time of the velocity $t_c(v) = 52ms$ (first zero), correlation time of the acceleration $t_c(a) = 1.5ms$ (first zero). The color of the curves correspond to the color of the integers $2n$ and $2n + 1$ written in the figures.

B. Lagrangian acceleration

One may ask if the method used above (to show up the presence of singular events), leads to the same results if the acceleration $a(t)$ is the Lagrangian acceleration,

$$a(r, t) = \partial v/\partial t + (v.\nabla)v, \tag{4}$$

measured by following the trajectories of small balls. To answer this question we have used the data of the von Karman experiment detailed in the thesis of N. Mordant and further papers[9]. Let us notice that the total number of points is only 3.2 millions, moreover they correspond to 9500 trajectories of balls having each a small mean time inside the recording area. Therefore it was not possible to derive test functions in two steps (averaging first inside each trajectory, and then averaging over the different trajectories that raises the problem of the weight to be attributed to each one). Therefore we have been constrained to join all trajectories in order to calculate the test functions. The result shown in Fig.2 displays qualitatively the same behavior as the one described above for the Eulerian acceleration in Modane's experiment, Fig.1.

III. IRREVERSIBILITY AND DISSIPATION

The observation of short distance "numerical" divergence of the high order (n large) structure functions was claimed above as a proof of the existence of singular events, however this does not settle the relevance of such singular events for dissipation in turbulent flows, the idea put forward by Taylor. It could be that singularities of solutions of the Euler inviscid equation only bounce so to speak at the time of the blow-up and so do not contribute at all or even significantly to dissipation. Indeed it could be that finite time singularities of the acceleration evolve similarly to what happens when an elastic ball bounces on a rigid plane. In this case its velocity changes in a very short time (as computed by the young Hertz [13]) between two opposed values, which yields an almost infinite acceleration but, to leading order, yields almost no dissipation (a small dissipation takes place by excitation of elastic vibration of the ball, but this is unimportant if the velocity of the ball is much less than the speed of sound in the solid, as assumed by Hertz). Therefore it is not unthinkable, by analogy with the bouncing ball, that a Leray-like

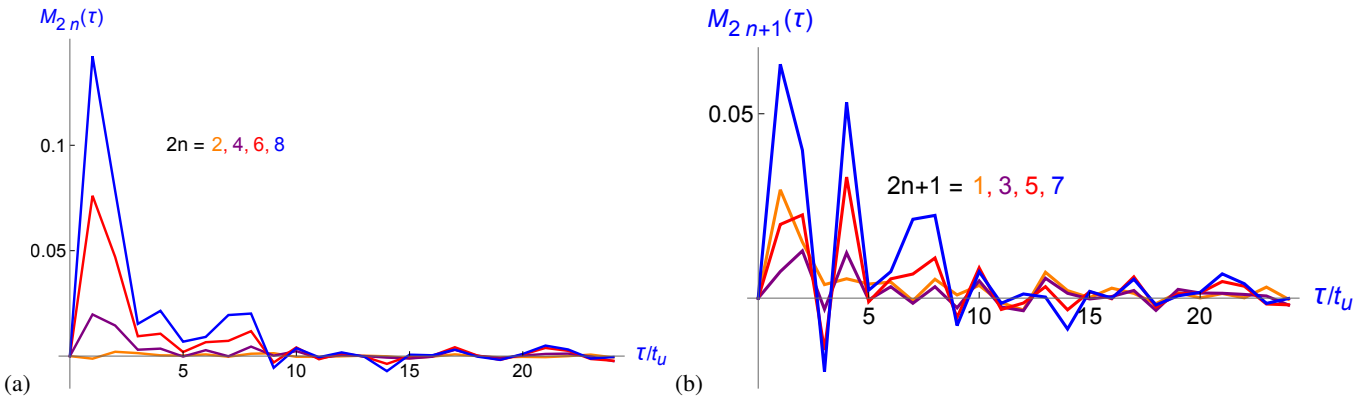


FIG. 3: Test functions of the acceleration, (a) M_{2n}, (b) M_{2n+1}, defined in Eq.(6) versus τ/t_u for $2n = 2, 4, 6, 8$ and $2n+1 = 5, 7$ respectively. The maxima increase with n. For $\tau = 0$ the test functions vanishe by definition. Same experimental data values as in Fig.1.

singularity is not dissipative or negligibly so and ends by bouncing back to the initial conditions with velocity reversed and so without dissipation. This kind of question (dissipation or not in singular events) is hard to answer on theoretical grounds because we do not have (yet) a thorough understanding of those singularities. Let us precise that singularities are not expected in the physical sense due to the discrete character of the structure of fluids at small scales (this is obvious in gas where the small scale is the mean free path which limits the motion of molecules, a property used by Maxwell to calculate the viscosity of dilute gas). Therefore we limit our study to spatial scales much larger than this discrete scale. In this frame we believe that our analysis of the structure functions at increasing exponents and small time difference is a strong argument in favor of the occurence of singularities of the Leray type in high Reynolds number flows, but this does not prove their existence in the mathematical sense! Therefore it seems reasonable to continue on the same track, namely to explore from the analysis of the hot wire records, and von Karman records, if the singularities are dissipative or not. Thinking to the bounce of a ball, this is dissipative if the velocity after the bounce is different of what it was before. Dissipation, not surprisingly, is put in evidence by the fundamentally irreversible character of the time evolution of the velocity: reversing time and velocities one does not get back the same history just because the velocity after the bounce is less than before. So by analyzing the experimental records in a way inspired by the one used to put in evidence the self-similar singularities we shall try to put in evidence also that the singularities are dissipative in the sense that they show a very strong irreversibility as the time difference is small and the amplitude of the fluctuation large. These large fluctuations being the ones found above, namely corresponding to increasing exponents in the definition of the structure functions.

In 1982, the paper[14] on fluctuations in out of equilibrium system explained that, because of the irreversibility, dissipative fluctuations break the time reversal symmetry, so that a whole set of test functions built with correlation functions of the time dependent fluctuations do not vanish, although they do in systems with time reversal symmetry. This paper gives a precise definition of irreversibility in turbulent flows, related to the hypothesis of cascade (studied in [15]) from large spatial scales to small ones where dissipation takes place. More recently the phenomena of energy transfer and dissipation, which imply both irreversibility, were considered [16]-[17]-[18]-[19] outside its connection with singularities. Our idea is to investigate those two phenomena in connection with singular events. In this view, the irreversibility associated to transfer of energy by cascading should be associated with a long time scale, of order the correlation time of the velocity, although the signal of irreversibility associated to dissipation should have a much shorter short time duration, as explained below.

Consider a time dependent signal $a(t)$ with fluctuations steady on average. A time correlation function built from $a(t)$ by taking averages like $< a(t)a(t+\tau) >$ with t integrated on a large interval of time, and τ fixed, is unchanged by time reversal and is an even function of τ. Consider now test functions also built up from $a(t)$ like

$$M_{m,n}(\tau) =< a^m(t)(a^n(t+\tau) - a^n(t+2\tau))a^m(t+3\tau) > , \tag{5}$$

where m and n are two positive integers. Under time reversal symmetry, this function changes sign for a non reversible fluctuations and it cancels to zero for a time symmetric system. By time symmetric we imply that no analysis of time-averages can yield a direction of time, which is true for fluctuations in system at equilibrium. If there is dissipation, there is no symmetry under time reversal. By dissipation we mean that part of the kinetic energy of the fluid motion is transformed into heat. For turbulent flows this is familiar concept of course, but our analysis shows that irreversibility of the fluctuations linked to the acceleration is present only at very short time scales.

A. Modane experiment

Using the time series recorded in the wind tunnel of Modane, we have studied the set of test functions $M_{1,n}(\tau)$ indexed by a single integer, and normalized in this form

$$M_n(\tau) = \frac{< a(t)(a^n(t+\tau) - a^n(t+2\tau))a(t+3\tau) >}{< a(t)^{n+2} >} \tag{6}$$

As shown in Fig.3-(a) the functions $M_{2n}(\tau)$ display 3 decreasing maxima close to the origin of time, each one of width $3t_u$, and vanish outside (up to the noise due to limited recording time, see captions for the parameters). For odd values of the integer, its behavior is more chaotic, see Fig.3-(b), and the amplitude of the irreversible signal is globally smaller, but in both cases the test functions are non zero precisely in the range of times where fluctuations are very big, namely where the structure functions display a diverging-like behavior at large order n. We consider this result as the proof of existence of dissipation resulting of the occurrence of finite time singularities of solutions of the Euler equations. Let us precise that we did analyse the long time behavior (not shown here) of these acceleration test functions which display no contribution at all of the irreversible character outside the small time domain of τ. We deduce that these test functions displays the irreversible character of the dissipation occurring mainly very close to the formation of singularities (after the stage of transfer of energy across the length scales during which the singular solutions are in formation). Practically one observes that the time domain where irreversibility -and- dissipation manifests, extends on an interval of order the correlation length of the acceleration, shown in Fig.3-(c) defined as

$$C_x(\tau) =< x(t)\ x(t+\tau) > . \tag{7}$$

with $x(t) = a(t)$ here.

Let us compare these results with our previous study of irreversibility which was performed on the velocity dynamics[15]. In this study we used the simple test function

$$\Psi_2(\tau) =< v^2(t)\ v(t+\tau) > - < v(t)\ v^2(t+\tau) > . \tag{8}$$

We found that this function built on the velocity $v(t)$ is non-vanishing on a much longer time interval that the one observed in the present study of the acceleration momenta. To make the story short, one observes that the duration of the "irreversible signal" is approximately equal to the correlation time $\tau_c^{(v)}$ of the velocity in [15] (see see figs. 3 and 4 in this paper where $\tau_c^{(v)} \approx 1sec$, whereas the duration of the irreversible signal in Figs.3 is of order of the correlation time of the acceleration, that lasts only $\tau_c^{(a)} \approx 4ms$, see Figs.3-c. Therefore the duration of the irreversible signal depends on the quantity we are looking for, it is approximately equal to the correlation time of the variable $x(t)$ considered in test function. In the present case the large difference between the correlation times of the velocity and acceleration is understandable because the numerical calculation of $C_x(\tau)$ by gliding time average defined in Eq.(7) amounts to make a convolution

$$C_x(\tau) = (x * x)(\tau). \tag{9}$$

that gives in the Fourier space

$$\hat{C}_a(\omega) = \omega^2 \hat{C}_v(\omega). \tag{10}$$

Therefore the role of the high frequencies (short time scales) is amplified when the velocity $v(t)$ is replaced by the acceleration $a(t)$. Our study proves that when taking $x(t) = v(t)$ the irreversibility manifests over a much longer part of the singular solution, although when taking $x(t) = a(t)$ the irreversible signal displays only the ultimate stage of the dynamics, the one where dissipation takes place.

In summary one is led to conclude that the irreversibility displayed in Figs.3 which is limited to the ultimate stage of events where the acceleration strongly increases, is the one envisioned by Taylor. We interpret the long dissipative time $\tau_c(v)$ yet observed in the study of velocity tests, as a signature of dissipation taking place during the whole formation of a singular event.

We emphasize that this interpretation of dissipation via singular events contradicts the hypothesis of a more or less uniform dissipation occurring after a smooth transfer of energy from large scale to smaller scales.

B. von Karman experiment

As above, we have used the data of the von Karman experiment, and have proceeded in the same way, i.e. averaging in a single step on the trajectories joined one after the other. The result is shown in figs.4. The amplitude of the test functions for

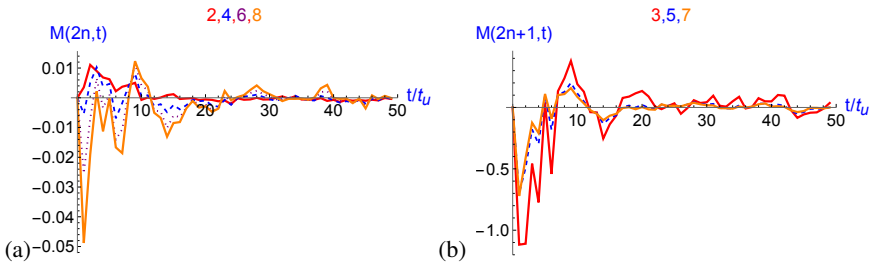


FIG. 4: (a) Test functions $M_{2n}(\tau)$ and , (b) $M_{2n+1}(\tau)$. The color of the curves correspond to the color of the integer $2n$ and $2n+1$ written in the figures.

odd numbers ($2n+1 = 3, 5, 7$) in Fig.(b) are noticeably larger than the amplitude in Fig.(a) for even values $2n = 2, 4, 6, 8$, that differs from the case of Modane's curves where the curves have comparable amplitudes. However one observes that, as in Modane's experiment, the width of the test functions is of the same order of magnitude as the width of the correlation function $C(\tau)$ shown in Fig.2-c. In summary one retrieves the important result detailed above concerning the dissipation, namely that dissipation occurs on a short time scale, of the order of the correlation time of the acceleration. This result agrees with dissipative time scale deduced in [16] (which is a purely spatial study, although our method is dynamic)

IV. CONCLUSION

We have extended the analysis of the records of the time fluctuations of the Eulerian acceleration, as given by hot wire records, completed by the records of the time fluctuations of the Lagrangian acceleration in Mordant's von Karman experiment. We have shown that the study of the structure functions $S_n(\tau)$ is fully in agreement with the existence of singular Leray-like events flying by the hot wire, or in the middle plane of the von Karman experimental set-up. Then we extended the multi-time correlation study to the problem of dissipation in such events via test functions M_n. We think that this makes a significant step forward in our understanding of fluid mechanical turbulence, because it allows to put in evidence that such singularities display a fundamentally irreversible character, in full agreement with the property that such fluctuations contribute to dissipation.

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